Chapter 3 Derivatives

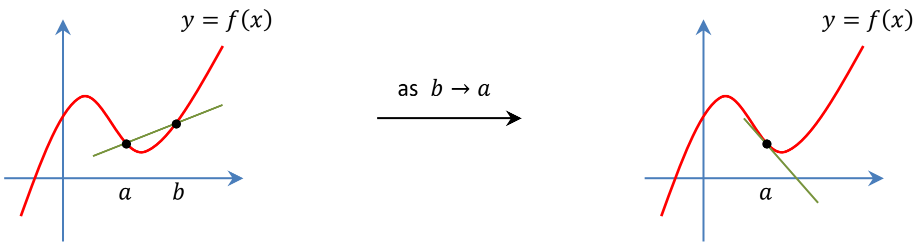
1. Derivatives and differentiability

Remark 3.1 Let f be a function defined on an interval, and let $a < b$ be two distinct real numbers in that interval. Let's consider the quotient

$$\frac{\boldsymbol{f(b) - f(a)}}{\boldsymbol{b - a}}.$$

- In geometry, this quotient represents the **slope of the "secant line"** joining the two points $(a, f(a))$ and $(b, f(b))$ on the graph of f.
- In science, this quotient represents the **average rate of change** of a physical quantity $f(x)$ as the value of the independent variable x gradually increases from a to b.
- In kinematics, if $f(t)$ specifically represents the position of a moving particle at t seconds after some initial time, then this quotient is the **average velocity** of the particle from the a^{th} second to the b^{th} second.

We aim to study the limit of the above quotient when a and b becomes so close to each other that the line joining the points $(a, f(a))$ and $(b, f(b))$ on the graph of f (a ***secant line***) becomes a line which touches the graph of f at the point $(a, f(a))$ (a ***tangent line***).



Definition 3.2 Let f be a function.

- Let a be a real number. The ***derivative of f at a*** is defined by the limit
$$f'(a) := \lim_{b \to a} \frac{f(b) - f(a)}{b - a}, \qquad \text{or equivalently } f'(a) := \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$
(obtained with a change of variable $h = b - a$).
- Replacing the fixed real number a by a real variable x, we say that the ***derivative of f*** is the function f' defined by
$$f'(x) := \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$
for every x in the domain of f at which the limit exists as a (finite) real number. The domain of this function f' is therefore a part of the domain of f.

Note that in general, the derivative of a function f at a number a **may or may not exist**. So we make the following definition.

Definition 3.3 Let f be a function.

- Let a be a real number. f is said to be ***differentiable at*** $\boldsymbol{a}$ if its derivative at a exists, i.e. the limit
$$\lim_{b \to a} \frac{f(b) - f(a)}{b - a}, \qquad \text{or} \qquad \lim_{h \to 0} \frac{f(a+h) - f(a)}{h},$$
exists as a finite real number.
- f is said to be ***differentiable on an interval*** if it is differentiable at every number in that interval.
- The process of finding the derivative of a function is called ***differentiation***. So to ***differentiate*** the function f means to find its derivative f', i.e. to compute $f'(x)$ for each x in the domain of f.

Remark 3.4 Let a be a real number and let f be a function which is differentiable at a. Then the graph of f is smooth near a.

- In geometry, $f'(a)$ represents the **slope of the tangent line** to the graph of f at the point $\big(a, f(a)\big)$.
- In science, $f'(a)$ represents the **"instantaneous" rate of change** of the physical quantity $f(x)$ when the value of the independent variable x is exactly a.
- In kinematics, if $f(t)$ represents the position of a moving particle at t seconds after an initial time, then $f'(a)$ is the **"instantaneous" velocity** of the particle at the a^{th} second.

Example 3.5 Let f be the function defined by
$$f(x) = x^2.$$
Find the derivative of f from definition.

Solution:

For every real number x, we have
$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} \\ &= \lim_{h \to 0} \frac{2hx + h^2}{h} = \lim_{h \to 0} (2x + h) \\ &= 2x. \end{aligned}$$

Example 3.6 Let f be the function defined by

$$f(x) = \frac{1}{x}.$$

Find the derivative of f from definition.

Solution:

For each non-zero real number x, we have

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\frac{1}{x+h}-\frac{1}{x}}{h}$$

$$= \lim_{h\to 0}\frac{x-(x+h)}{hx(x+h)} = \lim_{h\to 0}\frac{-1}{x(x+h)} = -\frac{1}{x^2}.$$

Example 3.7 Let f be the function defined by

$$f(x) = \sqrt{x}.$$

Find the derivative of f from definition.

Solution:

This does not work for $x = 0$.

For each **positive** real number x, we have

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}$$

$$= \lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}\cdot\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}$$

$$= \lim_{h\to 0}\frac{(x+h)-x}{h\left(\sqrt{x+h}+\sqrt{x}\right)} = \lim_{h\to 0}\frac{1}{\sqrt{x+h}+\sqrt{x}} = \frac{1}{2\sqrt{x}}.$$

Remark 3.8 In science and engineering applications, we usually denote by y a quantity whose value depends on an independent variable x, so that $y = f(x)$ for some function f. In this situation, the definition of the derivative of f

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

can be interpreted as follows. We rename the denominator h as Δx, which represents a "small change" in the independent variable x; and we rename the numerator $f(x+\Delta x)-f(x)$ as Δy, which represents the change in the quantity y that corresponds to this small change in x. Then

$$f'(x) = \lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}$$

is the limit of the average **rate of change of y with respect to x**.

Remark 3.9 Let f be a function and treat $y = f(x)$ as a quantity that depends on the independent variable x. Various different **notations** are often used to denote the derivative of a function f.

- (Lagrange) When we emphasize that the derivative of f is a function, we write f' or y'.
- (Leibniz) When we emphasize that the derivative of f is the rate of change of the quantity $y = f(x)$ with respect to the change in the independent variable x, we can also denote it by $\frac{dy}{dx}$. The derivative of f at a is then denoted by $\left.\frac{dy}{dx}\right|_{x=a}$.
- Sometimes we do not want to label the function by a symbol. For example, the derivative of $f(x) = x^2 + 3x$ can be written as $\frac{d}{dx}(x^2 + 3x)$. At a we write $\left.\frac{d}{dx}(x^2 + 3x)\right|_{x=a}$.

Example 3.5 Let $f: \mathbb{R} \to \mathbb{R}$ be the function defined by

$$f(x) = x^2.$$

We have already found that the derivative of f is $f'(x) = 2x$ for every $x \in \mathbb{R}$.

(a) Find the equation of the **tangent line** to the graph of f at the point $(3, f(3))$.

(b) For each real number a, find the equation of the **normal line** to the graph of f at the point $(a, f(a))$.

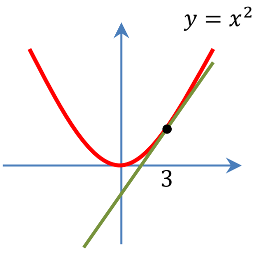
The ***normal line*** to a curve is the line that is perpendicular to the tangent line at the point of tangency.

Solution:

(a) At the point $(3, f(3)) = (3, 9)$, the slope of the tangent line to the graph of f is $f'(3) = 2(3) = 6$. Therefore the equation of the required tangent line is given by

$$y - 9 = 6(x - 3),$$

i.e. $y = 6x - 9$.



(b) At the point $(a, f(a)) = (a, a^2)$, the slope of the tangent line is $f'(a) = 2a$.

- If $a \neq 0$, then the slope of the normal line at the same point is $-\frac{1}{2a}$. So the equation of the normal line to the graph of f at (a, a^2) is given by

$$y - a^2 = -\frac{1}{2a}(x - a),$$

i.e. $x + 2ay - a - 2a^3 = 0$.

The product of the slopes of a pair of perpendicular lines is -1.

- If $a = 0$, then the tangent line to the graph of f is a horizontal line. So the normal line to the graph of f at $(0, 0)$ is a vertical line, and its equation is given by $x = 0$.

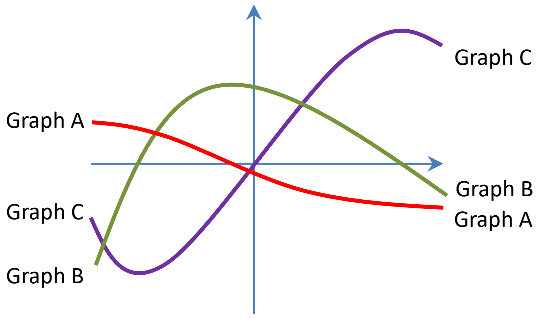
In both cases, the equation of the normal line to the graph of f at (a, a^2) is always given by

$$x + 2ay - a - 2a^3 = 0.$$

Example 3.10 The diagram below shows the graphs of three functions f, g and h such that

$$g = f' \qquad \text{and} \qquad h = g'.$$

Match each of these three functions with its corresponding graph and explain your choice.



Solution:

The derivative of a function represents the **slope of the tangent line** to the graph of the function. Observing that

⦿ graph B cuts the x-axis whenever the slope of the tangent line to graph C is horizontal, and

⦿ graph A cuts the x-axis whenever the slope of the tangent line to graph B is horizontal,

we conclude that C is the graph of f, B is the graph of g, and A is the graph of h.

Next, we recall that the differentiability of a function f at a real number a, i.e. the existence of the limit

$$f'(a) \coloneqq \lim_{h \to 0} \frac{f(a+h) - f(a)}{h},$$

requires that the one-sided limits

$$\lim_{h \to 0^-} \frac{f(a+h) - f(a)}{h} \qquad \text{and} \qquad \lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h}$$

both exist and are equal. We use the following quick notations for these **one-sided derivatives**.

Definition 3.11 Let a be a real number and f be a function. The one-sided limits

$$f'_-(a) \coloneqq \lim_{h \to 0^-} \frac{f(a+h) - f(a)}{h} \qquad \text{and} \qquad f'_+(a) \coloneqq \lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h}$$

are respectively called the ***left-hand derivative*** and the ***right-hand derivative*** of f at a.

Theorem 3.12 *Let a be a real number and f be a function. Then f is differentiable at a if and only if $f'_-(a)$ and $f'_+(a)$ both exist as finite real numbers and*

$$f'_-(a) = f'_+(a).$$

Example 3.13 Let $f: \mathbb{R} \to \mathbb{R}$ be the absolute value function

$$f(x) = |x|.$$

At which real numbers is f differentiable?

Solution:

- ⊙ For each $a > 0$, we have
$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0} \frac{|a+h|-|a|}{h} = \lim_{h\to 0} \frac{(a+h)-a}{h} = 1$$
which is a finite real number, so f is differentiable at each $a > 0$, with $f'(a) = 1$.
- ⊙ Similarly, for each $a < 0$, we have
$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0} \frac{|a+h|-|a|}{h} = \lim_{h\to 0} \frac{-(a+h)+a}{h} = -1$$
which is also a finite real number, so f is differentiable at each $a < 0$, with $f'(a) = -1$.
- ⊙ Finally, at 0 we have
$$f'_-(0) = \lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^-} \frac{|h|-|0|}{h} = \lim_{h\to 0^-} \frac{-h-0}{h} = -1, \quad \text{but}$$
$$f'_+(0) = \lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^+} \frac{|h|-|0|}{h} = \lim_{h\to 0^+} \frac{h-0}{h} = 1.$$
Since $f'_-(0) \neq f'_+(0)$, f is not differentiable at 0.

Therefore f is differentiable at every non-zero real number, i.e. on $(-\infty, 0) \cup (0, +\infty)$. To be precise, we have found that

$$f'(x) = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \end{cases}.$$

Theorem 3.14 *Let a be a real number and f be a function. If f is differentiable at a, then f is continuous at a.*

Proof. For every x near a and $x \neq a$, we have

$$f(x) = f(a) + [f(x) - f(a)] = f(a) + \frac{f(x)-f(a)}{x-a} \cdot (x-a).$$

Since f is differentiable at a, the limit $f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$ exists as a finite real number. So,

$$\begin{aligned} \lim_{x\to a} f(x) &= f(a) + \lim_{x\to a} \left[\frac{f(x)-f(a)}{x-a} \cdot (x-a) \right] \\ &= f(a) + \lim_{x\to a} \frac{f(x)-f(a)}{x-a} \cdot \lim_{x\to a} (x-a) \\ &= f(a) + f'(a) \cdot (a-a) = f(a). \end{aligned}$$

Therefore f is continuous at a. ∎

Theorem 3.14 is graphically intuitive; if the graph of a function is smooth, then the graph can be drawn without lifting the pen from the paper. With the help of this theorem, when we are given the differentiability of a function, we can automatically use its continuity as well.

Example 3.15 Let

$$f(x) = \begin{cases} 4x+1 & \text{if } x < 2 \\ a & \text{if } x = 2 \\ bx^2 + c & \text{if } x > 2 \end{cases}.$$

If f is differentiable at 2, find the values of a, b and c.

Solution:

Since f is differentiable at 2, it is continuous at 2 in particular. Thus we have

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) = f(2).$$

Now we have

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (4x+1) = 4(2)+1 = 9,$$

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (bx^2+c) = b(2)^2 + c = 4b + c,$$

$$f(2) = a.$$

These together imply that $a = 9 = 4b + c$.

Next we use the differentiability of f at 2, i.e. the existence of $\lim_{x\to 2} \frac{f(x)-f(2)}{x-2}$. We have

$$f'_-(2) = \lim_{x\to 2^-} \frac{f(x)-f(2)}{x-2} = \lim_{x\to 2^-} \frac{(4x+1)-9}{x-2} = \lim_{x\to 2^-} \frac{4x-8}{x-2} = \lim_{x\to 2^-} 4 = 4$$

and

$$f'_+(2) = \lim_{x\to 2^+} \frac{f(x)-f(2)}{x-2} = \lim_{x\to 2^+} \frac{(bx^2+c)-(4b+c)}{x-2}$$

$$= \lim_{x\to 2^+} \frac{b(x^2-4)}{x-2} = \lim_{x\to 2^+} b(x+2) = 4b.$$

So the existence of the limit implies that $4b = 4$. Therefore $a = 9$, $b = 1$ and $c = 5$.

Remark 3.16 Let a be a real number and f be a function such that f is continuous at a and the one-sided derivatives $f'_-(a)$ and $f'_+(a)$ both exist. Geometrically,

- $f'_-(a)$ represents the **slope of tangent line** to the graph of f **from the left** of the point $(a, f(a))$; while
- $f'_+(a)$ represents the **slope of tangent line** to the graph of f **from the right** of $(a, f(a))$.

Example 3.17 Sketch the graph of a function f which satisfies all the following conditions:

(i) The domain of f is $[0,5]$.

(ii) f is continuous on $[0,5]$.

(iii) $f'(x) < 0$ for every $x < 1$.

(iv) $\displaystyle\lim_{x\to 1^+}\frac{f(x)+1}{x-1} = 2$.

(v) $f'(3) = 0$.

(vi) $f'(x) < 0$ for every $x > 3$.

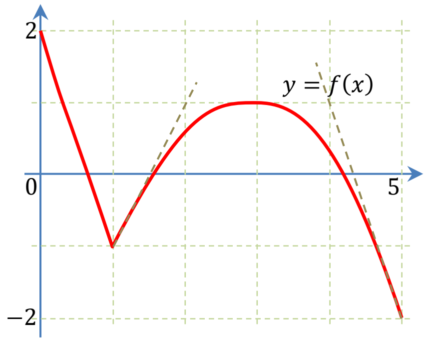
(vii) $\displaystyle\lim_{x\to 5^-}\frac{f(x)-f(5)}{x-5} = -3$.

Solution:

We need to translate the given information about the function f to information about its **graph**.

- Conditions (i) – (ii) imply that the graph is one single piece of curve, whose leftmost point hits $x = 0$ and rightmost point hits $x = 5$. Of course the curve needs to pass the vertical line test.
- Condition (iii) implies that on the interval $[0,1]$, the graph of f is smooth and the slope of its tangent line is negative.
- The limit in condition (iv) being equal to a finite real number implies that $\displaystyle\lim_{x\to 1^+} f(x) = -1$ (similar to Example 2.18). Since f is continuous at 1 by (ii), we must have $f(1) = -1$, i.e. the graph of f passes through the point $(1,-1)$. Thus in fact the limit in condition (iv) is $f'_+(1)$, which implies that the tangent line to the graph from the right of the point $(1,-1)$ has its slope equal to 2.
- Condition (v) implies that the graph of f has a horizontal tangent line near $x = 3$. It does not mention anything about the value $f(3)$ though.
- Condition (vi) implies that on the interval $[3,5]$, the graph of f is smooth and the slope of its tangent line is negative.
- The limit in condition (vii) is $f'_-(5)$, which implies that the tangent line to the graph from the left of the point $(5, f(5))$ has its slope equal to -3.

The graph of a possible function f that satisfies all the above conditions is shown below.



Remark 3.18 Note that a **continuous** function **may not be differentiable** at a point.

⊙ An example is the function

$$f(x) = |x|,$$

which is continuous on $\mathbb{R}$ but non-differentiable at 0 because $f'_+(0) \neq f'_-(0)$, as we have seen in Example 3.13. The slopes of tangent lines from the two sides of the point $(0,0)$ do not match, and the graph of f has a **corner** at $(0,0)$.

⊙ Another example is the function

$$f(x) = x^{\frac{1}{3}},$$

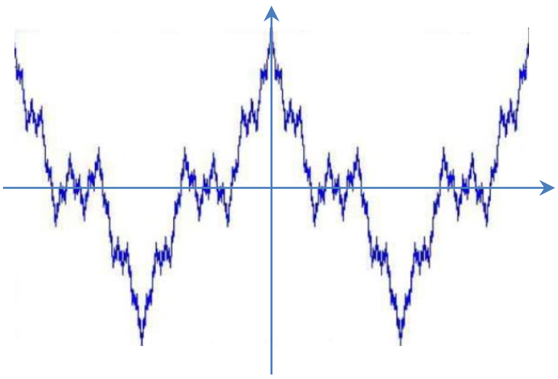
which is continuous on $\mathbb{R}$ but non-differentiable at 0 because

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{x^{\frac{1}{3}} - 0^{\frac{1}{3}}}{x - 0} = \lim_{x \to 0} \frac{1}{x^{\frac{2}{3}}} = +\infty.$$

The graph of f has a **vertical tangent line** at $(0,0)$.

⊙ In Example 3.20 on the next page, we will see a function which is again continuous on $\mathbb{R}$ but non-differentiable at 0, and has **no tangent line** there at all.

One can easily construct continuous functions that are non-differentiable at a discrete set of points. To the extreme case, there actually exist functions that are **everywhere continuous but nowhere differentiable**!



Remark 3.19 The logical relations between the following statements are important.

f is **differentiable** at a $f'_-(a) = f'_+(a)$ $\lim\limits_{x \to a} \frac{f(x)-f(a)}{x-a}$ exists and is finite	$\Rightarrow$	f is **continuous** at a $\lim\limits_{x \to a} f(x) = f(a)$	$\Rightarrow$ $\Rightarrow$	f is **well-defined** at a $f(a)$ exists $\lim\limits_{x \to a} f(x)$ exists

Example 3.20 Let $f, g\colon \mathbb{R} \to \mathbb{R}$ be the functions defined by

$$f(x) = \begin{cases} x^2 \sin\dfrac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} \qquad \text{and} \qquad g(x) = \begin{cases} x \sin\dfrac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}.$$

(a) Show that both f and g are continuous at 0.

(b) Show that f is differentiable at 0 but g is not differentiable at 0.

Proof:

(a) Since $-1 \leq \sin\frac{1}{x} \leq 1$ for every $x \neq 0$, we have

$$-x^2 \leq x^2 \sin\frac{1}{x} \leq x^2 \qquad \text{for every } x \neq 0.$$

Since $\lim\limits_{x\to 0} -x^2 = \lim\limits_{x\to 0} x^2 = 0$, according to Squeeze Theorem we have $\lim\limits_{x\to 0} x^2 \sin\frac{1}{x} = 0$. Thus

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} x^2 \sin\frac{1}{x} = 0 = f(0),$$

which shows that f is continuous at 0. In a similar fashion, we have

$$-|x| \leq x \sin\frac{1}{x} \leq |x| \qquad \text{for every } x \neq 0.$$

Since $\lim\limits_{x\to 0} -|x| = \lim\limits_{x\to 0} |x| = 0$, according to Squeeze Theorem we have $\lim\limits_{x\to 0} x \sin\frac{1}{x} = 0$. Thus

$$\lim_{x\to 0} g(x) = \lim_{x\to 0} x \sin\frac{1}{x} = 0 = g(0),$$

which shows that g is continuous at 0.

(b) We consider the definition of derivative at 0. Since

$$\lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{x^2 \sin\frac{1}{x} - 0}{x - 0} = \lim_{x\to 0} x \sin\frac{1}{x} = 0$$

according to (a), we see that f is differentiable at 0 and $f'(0) = 0$. On the other hand, we have

$$\lim_{x\to 0} \frac{g(x) - g(0)}{x - 0} = \lim_{x\to 0} \frac{x \sin\frac{1}{x} - 0}{x - 0} = \lim_{x\to 0} \sin\frac{1}{x}$$

which does not exist; so g is not differentiable at 0.

The function g above provides another example of a function that is **continuous but not differentiable** at a number. This time there is **no tangent line at $(0, 0)$** at all.

2. Rules of differentiation

Our next aim is to establish some **rules of differentiation** so that we can obtain a library of formulas of **derivatives of elementary functions**.

Theorem 3.21 *Let f and g be differentiable functions and c be a real number. Then*

$$(f+g)'=f'+g', \qquad (f-g)'=f'-g' \qquad \text{and} \qquad (cf)'=cf'.$$

Proof. We only prove the first equality. Recall that the function $f+g$ is defined by

$$(f+g)(x)=f(x)+g(x),$$

so for every x in the domain of $f+g$,

$$\begin{aligned}\lim_{h\to 0}\frac{(f+g)(x+h)-(f+g)(x)}{h} &= \lim_{h\to 0}\frac{[f(x+h)+g(x+h)]-[f(x)+g(x)]}{h}\\ &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}+\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\\ &= f'(x)+g'(x).\end{aligned}$$

Therefore $f+g$ is differentiable and $(f+g)'=f'+g'$. ■

Theorem 3.22 (Product rule and quotient rule) *Let f and g be differentiable functions. Then*

$$(fg)'=f'g+fg' \qquad \text{and} \qquad \left(\frac{f}{g}\right)'=\frac{f'g-fg'}{g^2}.$$

Note again that in the second equality, the domain of $\left(\frac{f}{g}\right)'$ excludes all those x with $g(x)=0$.

Proof. To prove the product rule, we recall that the function fg is defined by

$$(fg)(x)=f(x)g(x),$$

so for every x in the domain of fg,

$$\begin{aligned}\lim_{h\to 0}\frac{(fg)(x+h)-(fg)(x)}{h} &= \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h}\\ &= \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x+h)+f(x)g(x+h)-f(x)g(x)}{h}\\ &= \lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}g(x+h)+f(x)\frac{g(x+h)-g(x)}{h}\right]\\ &= \left[\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\right]\left[\lim_{h\to 0}g(x+h)\right]+f(x)\left[\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\right]\\ &= f'(x)g(x)+f(x)g'(x).\end{aligned}$$

Therefore fg is differentiable and $(fg)'=f'g+fg'$.

$\lim_{h\to 0}g(x+h)=g(x)$ follows from the continuity of g.

To prove the quotient rule, we observe that $f = \frac{f}{g} g$. So applying the product rule we have

$$f' = \left(\frac{f}{g}\right)' g + \frac{f}{g} g',$$

and rearranging this we obtain

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}.$$

■

Example 3.23 Find the derivative of the function

$$f(x) = \frac{1}{x\sqrt{x}}.$$

Solution:

Note that $f(x) = \frac{\frac{1}{x}}{\sqrt{x}}$ for every $x > 0$. We have seen in Examples 3.6 and 3.7 that the derivative of $\frac{1}{x}$ is $-\frac{1}{x^2}$ and the derivative of $\sqrt{x}$ is $\frac{1}{2\sqrt{x}}$. So by quotient rule we have for every $x > 0$,

$$f'(x) = \frac{\left(-\frac{1}{x^2}\right)\left(\sqrt{x}\right) - \left(\frac{1}{x}\right)\left(\frac{1}{2\sqrt{x}}\right)}{\left(\sqrt{x}\right)^2} = \frac{-x^{-\frac{3}{2}} - \frac{1}{2}x^{-\frac{3}{2}}}{x} = -\frac{3}{2}x^{-\frac{5}{2}}.$$

Example 3.24 Let f and g be two functions such that

$$f(1) = -1, \qquad g(1) = 2, \qquad f'(1) = 0 \qquad \text{and} \qquad g'(1) = 4.$$

Compute the derivative

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)} + \frac{g(x)}{f(x)}\right)\bigg|_{x=1}.$$

Put $x = 1$ **after** differentiating the function $\frac{f}{g} + \frac{g}{f}$.

Solution:

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)} + \frac{g(x)}{f(x)}\right) = \frac{d}{dx}\frac{f(x)}{g(x)} + \frac{d}{dx}\frac{g(x)}{f(x)}$$
$$= \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2} + \frac{g'(x)f(x) - g(x)f'(x)}{[f(x)]^2}.$$

At $x = 1$, we have

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)} + \frac{g(x)}{f(x)}\right)\bigg|_{x=1} = \frac{f'(1)g(1) - f(1)g'(1)}{[g(1)]^2} + \frac{g'(1)f(1) - g(1)f'(1)}{[f(1)]^2}$$
$$= \frac{(0)(2) - (-1)(4)}{2^2} + \frac{(4)(-1) - (2)(0)}{(-1)^2} = -3.$$

Now we make use of the rules to obtain **differentiation formulas** for various elementary functions. Let's start with some **power functions** and **trigonometric functions** first.

Theorem 3.25 *Let c be a real number. Then the derivative of the* ***constant function*** $f(x) = c$ *is the constant function* $f'(x) = 0$. *In Leibniz's notation this is written as*

$$\frac{d}{dx}c = 0.$$

Proof. Given $f(x) = c$ for every real number x, we have

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} = \lim_{h\to 0}\frac{c - c}{h} = \lim_{h\to 0} 0 = 0$$

for every real number x. ■

Theorem 3.25 is a special case of the following theorem.

Theorem 3.26 *Let n be an integer. Then*

$$\frac{d}{dx}x^n = nx^{n-1}.$$

In Lagrange's notation this means that the derivative of $f(x) = x^n$ is $f'(x) = nx^{n-1}$.

Idea of proof. The case when $n = 0$ is just Theorem 3.25. Suppose that n is a positive integer. For simplicity, let us illustrate the reasoning for the specific case when $n = 3$, and the case when n is an arbitrary positive integer is left as an exercise for interested readers. When $n = 3$, we have

$$\begin{aligned}\frac{d}{dx}x^n = \frac{d}{dx}x^3 = \lim_{h\to 0}\frac{(x+h)^3 - x^3}{h} &= \lim_{h\to 0}\frac{(x^3 + 3x^2h + 3xh^2 + h^3) - x^3}{h}\\ &= \lim_{h\to 0}\frac{3x^2h + 3xh^2 + h^3}{h} = \lim_{h\to 0}(3x^2 + 3xh + h^2)\\ &= 3x^2 + 0 + 0 = 3x^2\\ &= nx^{n-1}.\end{aligned}$$

Next, when n is a negative integer, say $n = -3$, we have

$$\begin{aligned}\frac{d}{dx}x^n = \frac{d}{dx}x^{-3} = \frac{d}{dx}\frac{1}{x^3} &= \frac{\left(\frac{d}{dx}1\right)(x^3) - (1)\left(\frac{d}{dx}x^3\right)}{(x^3)^2}\\ &= \frac{0\cdot x^3 - 1\cdot 3x^2}{x^6} = -3x^{-4}\\ &= nx^{n-1}\end{aligned}$$

as well. ■

In fact Theorem 3.26 holds even when n is **any real number**. We will deduce this fact later.

Example 3.27 Evaluate the derivatives of the following functions.

(a) $f(x) = (x^2 - 3x + 1)(x + 2) + x^{1013}$

(b) $g(t) = \dfrac{2t^2 + 5t + 3}{t + 2}$

Solution:

(a) We apply the product rule for the first term. For every $x \in \mathbb{R}$, we have

$$\begin{aligned} f'(x) &= \left\{\left[\frac{d}{dx}(x^2 - 3x + 1)\right](x + 2) + (x^2 - 3x + 1)\left[\frac{d}{dx}(x + 2)\right]\right\} + \frac{d}{dx}x^{1013} \\ &= [(2x - 3)(x + 2) + (x^2 - 3x + 1)(1)] + 1013x^{1012} \\ &= 1013x^{1012} + 3x^2 - 2x - 5 \end{aligned}$$

(b) We may apply the quotient rule. For every $t \in \mathbb{R} \setminus \{-2\}$, we have

$$\begin{aligned} g'(t) &= \frac{\left[\frac{d}{dt}(2t^2 + 5t + 3)\right](t + 2) - (2t^2 + 5t + 3)\left[\frac{d}{dt}(t + 2)\right]}{(t + 2)^2} \\ &= \frac{(4t + 5)(t + 2) - (2t^2 + 5t + 3)(1)}{(t + 2)^2} = \frac{2t^2 + 8t + 7}{(t + 2)^2} \end{aligned}$$

Example 3.28 Find the equation(s) of the tangent line(s) to the graph of the function

$$f(x) = \frac{2}{x^3}$$

which is (are) parallel to the line $6x + y = 0$.

Solution:

Since the line $6x + y = 0$ has slope -6, we need to look for those real numbers x at which $f'(x) = -6$. Now

$$f'(x) = 2(-3)x^{-4} = -\frac{6}{x^4},$$

so $f'(x) = -6$ only when $x^4 = 1$, i.e. when $x = 1$ or $x = -1$. These correspond to the points $\big(1, f(1)\big) = (1, 2)$ and $\big(-1, f(-1)\big) = (-1, -2)$ on the graph of f.

The equation of the tangent line at $(1, 2)$ is given by $y - 2 = -6(x - 1)$, i.e.

$$6x + y - 8 = 0,$$

while the equation of the tangent line at $(-1, -2)$ is given by $y + 2 = -6(x + 1)$, i.e.

$$6x + y + 8 = 0.$$

Theorem 3.29 (Derivatives of trigonometric functions)

$$\frac{d}{dx}\sin x = \cos x \qquad \frac{d}{dx}\tan x = \sec^2 x \qquad \frac{d}{dx}\sec x = \sec x \tan x$$
$$\frac{d}{dx}\cos x = -\sin x \qquad \frac{d}{dx}\cot x = -\csc^2 x \qquad \frac{d}{dx}\csc x = -\csc x \cot x$$

Proof. To prove that $\frac{d}{dx}\sin x = \cos x$,

$$\frac{d}{dx}\sin x = \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h} = \lim_{h\to 0}\frac{2\cos\left(x+\frac{h}{2}\right)\sin\frac{h}{2}}{h}$$
$$= \lim_{h\to 0}\cos\left(x+\frac{h}{2}\right)\lim_{h\to 0}\frac{\sin\frac{h}{2}}{\frac{h}{2}} = \cos x\cdot 1 = \cos x.$$

The proof of $\frac{d}{dx}\cos x = -\sin x$ is similar, so we leave it as an exercise.

To prove that $\frac{d}{dx}\tan x = \sec^2 x$, we recall that $\tan x = \frac{\sin x}{\cos x}$ and make use of the quotient rule.

$$\frac{d}{dx}\tan x = \frac{d}{dx}\frac{\sin x}{\cos x} = \frac{\left(\frac{d}{dx}\sin x\right)(\cos x)-(\sin x)\left(\frac{d}{dx}\cos x\right)}{(\cos x)^2}$$
$$= \frac{(\cos x)(\cos x)-(\sin x)(-\sin x)}{(\cos x)^2} = \frac{\cos^2 x+\sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x.$$

The proofs of the three remaining formulas are similar, so we leave them as exercise too. (Do try them out on your own.) ■

Example 3.30 Find the derivative of the function

$$h(\theta) = \frac{\sec\theta\tan\theta}{\theta^3}.$$

Solution:

$$h'(\theta) = \frac{\left(\frac{d}{d\theta}\sec\theta\tan\theta\right)(\theta^3)-(\sec\theta\tan\theta)\left(\frac{d}{d\theta}\theta^3\right)}{(\theta^3)^2}$$
$$= \frac{[(\sec\theta\tan\theta)(\tan\theta)+(\sec\theta)(\sec^2\theta)](\theta^3)-(\sec\theta\tan\theta)(3\theta^2)}{\theta^6}$$
$$= \frac{\theta\sec\theta\tan^2\theta+\theta\sec^3\theta-3\sec\theta\tan\theta}{\theta^4}.$$

3. Chain rule

After studying rules of differentiation for the sum, the difference, the product and the quotient in the previous section, we now study an important rule of differentiation about the **composition**.

Theorem 3.31 (Chain rule) *Let x be a real number and let f and g be functions. If g is differentiable at x and f is differentiable at $g(x)$, then $f \circ g$ is also differentiable at x and its derivative is given by*

$$(f \circ g)'(x) = f'\big(g(x)\big)g'(x).$$

If we write $u = g(x)$ and $y = f(u) = (f \circ g)(x)$, then in Leibniz's notation this formula becomes

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

The following sloppy proof of the chain rule is enough to illustrate why the rule should hold. The rigorous proof contains some distracting technical details and is omitted.

A sloppy proof. Recalling that the function $f \circ g$ is defined by $(f \circ g)(x) = f\big(g(x)\big)$, we have

$$\begin{aligned}(f \circ g)'(x) &= \lim_{h\to 0}\frac{(f \circ g)(x+h) - (f \circ g)(x)}{h} \\ &= \lim_{h\to 0}\frac{f\big(g(x+h)\big) - f\big(g(x)\big)}{h} \\ &= \lim_{h\to 0}\frac{f\big(g(x) + g(x+h) - g(x)\big) - f\big(g(x)\big)}{g(x+h) - g(x)} \cdot \frac{g(x+h) - g(x)}{h} \\ &= \lim_{h\to 0}\frac{f\big(g(x) + g(x+h) - g(x)\big) - f\big(g(x)\big)}{g(x+h) - g(x)} \lim_{h\to 0}\frac{g(x+h) - g(x)}{h}.\end{aligned}$$

With the change of variable $t = g(x+h) - g(x)$, we have $t \to 0$ as $h \to 0$ since g is continuous. Therefore

$$(f \circ g)'(x) = \lim_{t\to 0}\frac{f(g(x) + t) - f\big(g(x)\big)}{t} \lim_{h\to 0}\frac{g(x+h) - g(x)}{h} = f'\big(g(x)\big)g'(x).$$

■

Example 3.32 Find the derivative of each of the following functions.

(a) $f(x) = \sin(x^2)$

(b) $g(x) = \sin^2 x = (\sin x)^2$

(c) $h(x) = \tan\dfrac{x^2 + 3}{x - 5}$

(d) $p(s) = \left(1 - \sqrt{s^2 + 1}\right)^3$

Solution:

[We can use either Lagrange's notation or Leibniz's notation to present the solution.]

Lagrange's notation

(a) Writing $v(u) = \sin u$ and $u(x) = x^2$, we have $f(x) = v(u(x))$. So

$$\begin{aligned} f'(x) &= v'(u)u'(x) \\ &= (\cos u)(2x) \\ &= 2x\cos(x^2). \end{aligned}$$

(b) Writing $v(u) = u^2$ and $u(x) = \sin x$, we have $g(x) = v(u(x))$. So

$$\begin{aligned} g'(x) &= v'(u)u'(x) \\ &= (2u)(\cos x) \\ &= 2\sin x\cos x. \end{aligned}$$

(c) $h(x)$ is the composition of the functions $v(u) = \tan u$ and $u(x) = \frac{x^2+3}{x-5}$. So

$$\begin{aligned} h'(x) &= v'(u)u'(x) \\ &= (\sec^2 u)\frac{(2x)(x-5)-(x^2+3)(1)}{(x-5)^2} \\ &= \frac{x^2-10x-3}{(x-5)^2}\sec^2\frac{x^2+3}{x-5}. \end{aligned}$$

Leibniz's notation

(a) Let $y = f(x) = \sin x^2$. Then we may write $y = \sin u$ where $u = x^2$. So

$$\begin{aligned} f'(x) = \frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx} \\ &= (\cos u)(2x) \\ &= 2x\cos(x^2). \end{aligned}$$

(b) Let $y = f(x) = (\sin x)^2$. Then we may write $y = u^2$ where $u = \sin x$. So

$$\begin{aligned} f'(x) = \frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx} \\ &= (2u)(\cos x) \\ &= 2\sin x\cos x. \end{aligned}$$

(c) Denote $y = h(x) = \tan[(x^2+3)/(x-5)]$. Then $y = \tan u$ where $u = \frac{x^2+3}{x-5}$. So

$$\begin{aligned} h'(x) &= \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} \\ &= (\sec^2 u)\frac{(2x)(x-5)-(x^2+3)(1)}{(x-5)^2} \\ &= \frac{x^2-10x-3}{(x-5)^2}\sec^2\frac{x^2+3}{x-5}. \end{aligned}$$

(d) p is also a composition of **three** elementary functions. Let $y = p(s) = \left(1-\sqrt{s^2+1}\right)^3$.

Then we may write $y = u^3$, where $u = 1-\sqrt{v}$ and $v = s^2+1$. So

$$\begin{aligned} p'(s) = \frac{dy}{ds} = \frac{dy}{du}\frac{du}{dv}\frac{dv}{ds} &= (3u^2)\left(-\frac{1}{2\sqrt{v}}\right)(2s) \\ &= \left[3\left(1-\sqrt{s^2+1}\right)^2\right]\left(-\frac{1}{2\sqrt{s^2+1}}\right)(2s) \\ &= \frac{-3s\left(1-\sqrt{s^2+1}\right)^2}{\sqrt{s^2+1}}. \end{aligned}$$

Example 3.33 The following table shows some values of the functions f, g, f' and g'.

t	-1	1	2
$f(t)$	1	-2	-1
$g(t)$	-5	-1	-2
$f'(t)$	-3	0	2
$g'(t)$	3	-4	0

Using information from the above table, evaluate the following derivatives.

(a) $\left.\dfrac{d}{dt}\left[t^2 g\big(f(t)\big)\right]\right|_{t=2}$

(b) $\left.\dfrac{d}{dt}\dfrac{f\big(g(t)\big)}{[f(t)]^2}\right|_{t=1}$

Solution:

(a)

$$\frac{d}{dt}\left[t^2 g\big(f(t)\big)\right] = (2t)g\big(f(t)\big) + (t^2)\left[g'\big(f(t)\big)f'(t)\right].$$

At $t = 2$, we have

$$\begin{aligned}\left.\frac{d}{dt}\left[t^2 g\big(f(t)\big)\right]\right|_{t=2} &= 2(2)g\big(f(2)\big) + (2^2)g'\big(f(2)\big)f'(2) \\ &= 4g(-1) + 4g'(-1)f'(2) \\ &= 4(-5) + 4(3)(2) \\ &= 4.\end{aligned}$$

Plug in the value of t **after** evaluating the derivative.

(b)

$$\frac{d}{dt}\frac{f\big(g(t)\big)}{[f(t)]^2} = \frac{\left[f'\big(g(t)\big)g'(t)\right][f(t)]^2 - \left[f\big(g(t)\big)\right][2f(t)f'(t)]}{[f(t)]^4}.$$

At $t = 1$, we have

$$\begin{aligned}\left.\frac{d}{dt}\frac{f\big(g(t)\big)}{[f(t)]^2}\right|_{t=1} &= \frac{\left[f'\big(g(1)\big)g'(1)\right][f(1)]^2 - \left[f\big(g(1)\big)\right][2f(1)f'(1)]}{[f(1)]^4} \\ &= \frac{[f'(-1)\cdot(-4)](-2)^2 - [f(-1)][2(-2)(0)]}{(-2)^4} \\ &= \frac{[(-3)\cdot(-4)](-2)^2 - (1)[2(-2)(0)]}{(-2)^4} \\ &= 3.\end{aligned}$$

Example 3.34 Let f be a differentiable function such that

$$\frac{d}{dx}\left[f\left(\frac{1}{3}x^3\right)\right] = 2x^5.$$

Find $f'(x)$.

Solution: Let $u = \frac{1}{3}x^3$. Then $\frac{du}{dx} = \frac{1}{3}(3x^2) = x^2$. Now from the chain rule, we have

$$2x^5 = \frac{d}{dx}\left[f\left(\frac{1}{3}x^3\right)\right] = \frac{d}{dx}f(u) = f'(u)\cdot\frac{du}{dx} = f'(u)\cdot x^2,$$

so $f'(u) = 2x^3 = 2(3u) = 6u$. Replacing u by x, we get $f'(x) = 6x$.

Next let's compute the derivatives of **exponential functions**. Let $a > 0$ and let $f(x) = a^x$. Then for every real number x we have

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{a^{x+h}-a^x}{h} = \lim_{h\to 0}\frac{a^h-1}{h}\cdot a^x$$
$$= \underbrace{f'(0)}_{\text{a number}}\cdot a^x.$$

So the derivative of an exponential function is **proportional to itself**. In fact the limit

$$\lim_{h\to 0}\frac{a^h-1}{h}$$

exists for every positive number a, is continuous as a function of a, and is strictly increasing as a increases. Numerical approximations give

$$\lim_{h\to 0}\frac{2^h-1}{h} \approx 0.693 < 1 \qquad \text{and} \qquad \lim_{h\to 0}\frac{3^h-1}{h} \approx 1.099 > 1,$$

so by Intermediate Value Theorem, there must exist a (unique) number $e \in (2,3)$ such that

$$\lim_{h\to 0}\frac{e^h-1}{h} = 1.$$

Definition 3.35 ***Euler's number*** e is defined to be the number such that

$$\lim_{h\to 0}\frac{e^h-1}{h} = 1.$$

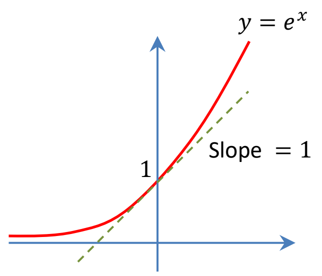
Remark 3.36 The base e of the natural exponential function

$$f(x) = e^x$$

is specially chosen so that the slope of tangent to the graph at the point $(0,1)$ is exactly

$$f'(0) = 1.$$

This is the reason why we call it "natural".



The discussion in the previous page already proves the following theorem.

Theorem 3.37 (Derivative of the natural exponential function)

$$\frac{d}{dx}e^x = e^x.$$

Let's digress a little bit for something more about the number e.

Theorem 3.38

$$e = \lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^x = \lim_{x\to-\infty}\left(1+\frac{1}{x}\right)^x.$$

Proof. From Definition 3.35, we have the right-hand limit

$$\lim_{h\to 0^+}\frac{e^h-1}{h} = 1$$

in particular. With the change of variables $h = \ln\left(1+\frac{1}{x}\right)$, we get

$$1 = \lim_{x\to+\infty}\frac{\left(1+\frac{1}{x}\right)-1}{\ln\left(1+\frac{1}{x}\right)} = \lim_{x\to+\infty}\frac{1}{x\ln\left(1+\frac{1}{x}\right)} = \lim_{x\to+\infty}\frac{1}{\ln\left(1+\frac{1}{x}\right)^x}$$

which gives $\lim\limits_{x\to+\infty}\ln\left(1+\frac{1}{x}\right)^x = 1$. Therefore

The exponential function is continuous.

$$\lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^x = \lim_{x\to+\infty}e^{\ln\left(1+\frac{1}{x}\right)^x} = e^{\lim_{x\to+\infty}\ln\left(1+\frac{1}{x}\right)^x} = e^1 = e,$$

which proves the first limit. The second limit can be proved in a similar way by using the left-hand limit $h\to 0^-$ instead. ■

Applying a change of variable $t = \frac{1}{x}$ to Theorem 3.38, we immediately obtain the following.

Corollary 3.39

$$\lim_{x\to 0}(1+x)^{\frac{1}{x}} = e.$$

Example 3.40 Evaluate each of the following limits.

(a) $\displaystyle\lim_{x\to+\infty}\left(\frac{x+1}{x-1}\right)^x$

(b) $\displaystyle\lim_{x\to+\infty}\left(1-\frac{1}{x+1}\right)^{x^2}$

Solution:

(a) With the change of variable $t = \frac{x-1}{2}$, we have $x = 2t + 1$ and so

$$\begin{aligned}
\lim_{x\to+\infty}\left(\frac{x+1}{x-1}\right)^x &= \lim_{x\to+\infty}\left(1+\frac{2}{x-1}\right)^x = \lim_{t\to+\infty}\left(1+\frac{1}{t}\right)^{2t+1} \\
&= \left[\lim_{t\to+\infty}\left(1+\frac{1}{t}\right)^t\right]^2 \lim_{t\to+\infty}\left(1+\frac{1}{t}\right) \\
&= e^2 \cdot 1 = e^2.
\end{aligned}$$

(b) With the change of variable $t = -(x+1)$, we have $x = -t-1$ and so

$$\lim_{x\to+\infty}\left(1-\frac{1}{x+1}\right)^{x^2} = \lim_{t\to-\infty}\left(1+\frac{1}{t}\right)^{(-t-1)^2} = \lim_{t\to-\infty}\left(1+\frac{1}{t}\right)^{t^2+2t+1}.$$

Let's analyze the limit $\lim_{t\to-\infty}\left(1+\frac{1}{t}\right)^{t^2}$. Since $\lim_{t\to-\infty}\left(1+\frac{1}{t}\right)^t = e > 2$ exists, it follows that

$$\left(1+\frac{1}{t}\right)^t > 2$$

for every t with $-t$ sufficiently large (more precisely, the inequality holds for every $t < -1$). So for all these (negative) numbers t we have

$$0 < \left(1+\frac{1}{t}\right)^{t^2} = \left[\left(1+\frac{1}{t}\right)^t\right]^t < 2^t.$$

Since $\lim_{t\to-\infty} 2^t = 0$, according to the Squeeze Theorem we have

$$\lim_{t\to-\infty}\left(1+\frac{1}{t}\right)^{t^2} = 0.$$

Therefore

$$\begin{aligned}
\lim_{x\to+\infty}\left(1-\frac{1}{x+1}\right)^{x^2} &= \lim_{t\to-\infty}\left(1+\frac{1}{t}\right)^{t^2} \cdot \lim_{t\to-\infty}\left(1+\frac{1}{t}\right)^{2t} \cdot \lim_{t\to-\infty}\left(1+\frac{1}{t}\right) \\
&= 0 \cdot e^2 \cdot 1 = 0.
\end{aligned}$$

Theorem 3.41 (Derivatives of exponential functions) *Let* $a > 0$. *Then*

$$\frac{d}{dx}a^x = a^x \ln a.$$

Proof. The case when $a = 1$ is just Theorem 3.25. If $a \neq 1$, then we have $a^x = e^{(\ln a)x}$ for every $x \in \mathbb{R}$, so

$$\frac{d}{dx}a^x = \frac{d}{dx}e^{(\ln a)x} = e^{(\ln a)x}\ln a = a^x \ln a$$

by the chain rule. ∎

Example 3.42 Find the derivative of the function

$$f(t) = e^{t^2} 5^{\sin t}.$$

Solution:

$$\begin{aligned} f'(t) &= \left(\frac{d}{dt} e^{t^2}\right)\left(5^{\sin t}\right) + \left(e^{t^2}\right)\left(\frac{d}{dt} 5^{\sin t}\right) \\ &= \left(e^{t^2} \cdot 2t\right)\left(5^{\sin t}\right) + \left(e^{t^2}\right)\left(5^{\sin t} \ln 5 \cdot \cos t\right) \\ &= e^{t^2} 5^{\sin t}(2t + (\ln 5) \cos t). \end{aligned}$$

Combining Theorem 3.41 with the discussion in the middle of page 19, we obtain the following.

Corollary 3.43 *Let* $a > 0$. *Then*

$$\lim_{h \to 0} \frac{a^h - 1}{h} = \ln a.$$

4. Techniques of differentiation

As a consequence of the chain rule, we obtain some more **techniques of differentiation**:

⦿ Implicit differentiation

⦿ Logarithmic differentiation

The first technique will enable us to differentiate the **inverse** of a one-to-one differentiable function, so that we can establish the formulas of derivatives of the remaining elementary functions, i.e. the **logarithmic functions** and the **inverse trigonometric functions**.

Remark 3.44 Sometimes the relationship between two quantities x and y cannot be expressed explicitly in the form $y = f(x)$. Instead, they are related **implicitly** via an equation of the form

$$F(x, y) = 0.$$

As an example, in the equation of the unit circle centered at the origin, we have

$$x^2 + y^2 - 1 = 0$$

In this situation, y cannot be expressed as a function of x, because the unit circle fails the vertical line test. However, we can still "pretend that y is a function of x" and find dy/dx **in terms of** $\boldsymbol{x}$ **and** $\boldsymbol{y}$ using chain rule. This technique is called **implicit differentiation**.

⦿ Although the circle fails the vertical line test, y can in fact be written as two functions of x:

$$y = \sqrt{1 - x^2} \qquad \text{or} \qquad y = -\sqrt{1 - x^2}.$$

The two functions correspond to the upper half and the lower half of the circle respectively. So apart from the two points $(1, 0)$ and $(-1, 0)$, it is alright to **"locally" regard** $\boldsymbol{y}$ **as one single function of** $\boldsymbol{x}$.

Example 3.45 Find $\frac{dy}{dx}$ if x and y are related implicitly by the equation

$$x^2 + y^2 + x\cos y = 1.$$

Solution:
Differentiating both sides of the equation **with respect to** $\boldsymbol{x}$, we have

$$2x + \frac{d}{dx}y^2 + \left[(1)(\cos y) + (x)\left(\frac{d}{dx}\cos y\right)\right] = 0.$$

Applying the chain rule, we have

$$2x + 2y\frac{dy}{dx} + \left[(1)(\cos y) + (x)(-\sin y)\left(\frac{dy}{dx}\right)\right] = 0,$$

which simplifies to

$$(2y - x\sin y)\frac{dy}{dx} = -2x - \cos y\,,$$

and so

$$\frac{dy}{dx} = \frac{2x + \cos y}{x\sin y - 2y}.$$

Example 3.46 Find the equation of the tangent line to the curve

$$3x^2y + y^3 = 4$$

at the point $(1, 1)$.

Solution:
Differentiating both sides of the given equation of the curve **with respect to** $\boldsymbol{x}$, we get

$$\left(6xy + 3x^2\frac{dy}{dx}\right) + 3y^2\frac{dy}{dx} = 0,$$

so

$$\frac{dy}{dx} = -\frac{2xy}{x^2 + y^2}.$$

So the slope of the tangent line at the point $(1, 1)$ is given by

$$\left.\frac{dy}{dx}\right|_{(1,1)} = -\frac{2(1)(1)}{1^2 + 1^2} = -1.$$

Therefore the equation of the tangent line at $(1, 1)$ is given by

$$y - 1 = -1(x - 1),$$

i.e. $x + y - 2 = 0.$

The chain rule or implicit differentiation enables us to find the derivative of the **inverse** of a one-to-one differentiable function.

Theorem 3.47 (Derivative of inverse) *Let f be a one-to-one differentiable function with inverse f^{-1}, and let y be in the range of f. If $f'\big(f^{-1}(y)\big) \neq 0$ and if f^{-1} is differentiable at y, then*

$$(f^{-1})'(y) = \frac{1}{f'\big(f^{-1}(y)\big)}.$$

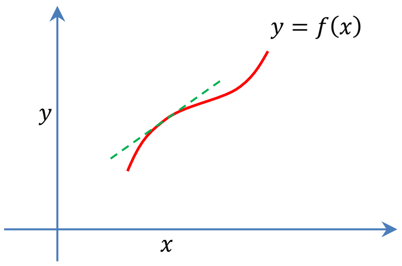
If we denote $y = f(x)$ and $x = f^{-1}(y)$, then the above formula can also be written as $(f^{-1})'(y) = 1 \,/\, f'(x)$, or in Leibniz's notation as

$$\frac{dx}{dy} = \frac{1}{dy/dx}.$$

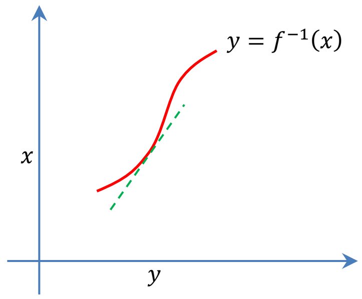
Proof. For every y in the range of f, we have $f\big(f^{-1}(y)\big) = y$. Differentiating both sides of this equation with respect to y and applying the chain rule, we have

$$f'\big(f^{-1}(y)\big) \cdot (f^{-1})'(y) = 1.$$

If $f'\big(f^{-1}(y)\big) \neq 0$, then we have $(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))}$. ■



Slope of tangent to the graph of f at x is m



Slope of tangent to the graph of f^{-1} at y is $\frac{1}{m}$

Example 3.48 Consider the function $f\colon \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = x^3 + x + 1.$$

It is given that f is one-to-one and f^{-1} is differentiable at 3. Compute $(f^{-1})'(3)$.

Solution: [It is too difficult to first find f^{-1} and then differentiate. Let's use Theorem 3.47.]
For every real number x we have

$$f'(x) = 3x^2 + 1.$$

Now we notice that $f(1) = 1^3 + 1 + 1 = 3$. Since f is one-to-one, it follows that $f^{-1}(3) = 1$. Thus by Theorem 3.47 we have

$$(f^{-1})'(3) = \frac{1}{f'\big(f^{-1}(3)\big)} = \frac{1}{f'(1)} = \frac{1}{3(1)^2 + 1} = \frac{1}{4}.$$

Theorem 3.49 (Derivatives of logarithms) *Let* $a \neq 1$ *be a positive number. Then*

$$\frac{d}{dx}\ln x = \frac{1}{x} \qquad \textit{and} \qquad \frac{d}{dx}\log_a x = \frac{1}{x\ln a}$$

as functions defined on $(0, +\infty)$.

Proof. Let $g(x) = \ln x$ for every $x > 0$ and $f(x) = e^x$. Since g is the inverse of f, we have

$$g'(x) = \frac{1}{f'\big(g(x)\big)} = \frac{1}{e^{g(x)}} = \frac{1}{e^{\ln x}} = \frac{1}{x}$$

for every $x > 0$. The second formula follows immediately from the first one when we note that

$$\log_a x = \frac{\ln x}{\ln a}$$

for every $x > 0$. ■

Remark 3.50 Note that the formulas in Theorem 3.49 are valid on the interval $(0, +\infty)$ only. On the interval $(-\infty, 0)$, we have

$$\frac{d}{dx}\ln(-x) = \frac{1}{x}$$

instead, according to the chain rule. Therefore we may combine this with Theorem 3.49 and write

$$\frac{d}{dx}\ln|x| = \frac{1}{x}$$

which is valid on $(-\infty, 0) \cup (0, +\infty)$ altogether.

We mentioned before that Theorem 3.26 holds even when n is **any real number**. Now we can prove this fact using the chain rule.

Theorem 3.51 (Power rule) *Let* r *be a real number. Then*

$$\frac{d}{dx}x^r = rx^{r-1}.$$

Proof. Note that $x^r = e^{r\ln x}$ for every $x > 0$. So by the chain rule we have

$$\frac{d}{dx}x^r = \frac{d}{dx}e^{r\ln x} = e^{r\ln x}\frac{r}{x} = x^r\frac{r}{x} = rx^{r-1}.$$

■

Think before proof: Why does each pair differ just by a minus sign?

Theorem 3.52 (Derivatives of inverse trigonometric functions)

$$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\arctan x = \frac{1}{1+x^2} \qquad \frac{d}{dx}\operatorname{arcsec} x = \frac{1}{|x|\sqrt{x^2-1}}$$

$$\frac{d}{dx}\arccos x = -\frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\operatorname{arccot} x = -\frac{1}{1+x^2} \qquad \frac{d}{dx}\operatorname{arccsc} x = -\frac{1}{|x|\sqrt{x^2-1}}$$

Proof. For every $x \in (-1,1)$, let $y = \arcsin x$. Then $y \in (-\pi/2, \pi/2)$ and $x = \sin y$, so

$$\frac{dx}{dy} = \cos y\,.$$

Therefore

$y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}.$$

The proofs of the remaining formulas are similar and are left as exercise. ■

Example 3.53 Find the derivatives of each of the following functions.

(a) $f(x) = \arccos\dfrac{1}{x}$

(b) $g(t) = \sin[\arctan(\ln t)]$

Solution:

(a) For every $x \in (-\infty,-1) \cup (1,+\infty)$, we have

$$f'(x) = -\frac{1}{\sqrt{1-(1/x)^2}} \cdot \left(-\frac{1}{x^2}\right) = \frac{1}{x^2\sqrt{1-1/x^2}} = \frac{1}{\sqrt{x^4-x^2}}.$$

(b) For every $t > 0$, we have

$$g'(t) = \cos[\arctan(\ln t)] \cdot \frac{1}{1+(\ln t)^2} \cdot \frac{1}{t} = \frac{\cos[\arctan(\ln t)]}{t[1+(\ln t)^2]}.$$

To find the derivatives of functions f involving exponents or indices, e.g. complicated exponential functions and algebraic functions, the method of **logarithmic differentiation** is sometimes useful.

Corollary 3.54 (Logarithmic differentiation) *Let f be a differentiable function. Then for every x at which $f(x) \neq 0$, we have*

$$\frac{d}{dx}\ln|f(x)| = \frac{f'(x)}{f(x)}.$$

Proof. Theorem 3.49 and chain rule. ■

Example 3.55 Find the derivative of the function

$$f(x) = \frac{x^{\frac{1}{3}}\sqrt{x^2+1}}{(3x+2)^3}.$$

Solution: [Of course this can be done by product rule and chain rule, but using logarithmic differentiation will save a lot of effort.]

For every $x \in \mathbb{R} \setminus \left\{0, -\frac{2}{3}\right\}$, consider

$$\ln|f(x)| = \ln\left|\frac{x^{\frac{1}{3}}\sqrt{x^2+1}}{(3x+2)^3}\right| = \ln\frac{|x|^{\frac{1}{3}}\sqrt{x^2+1}}{|3x+2|^3}$$

$$= \ln|x|^{\frac{1}{3}} + \ln\sqrt{x^2+1} - \ln|3x+2|^3$$

$$= \frac{1}{3}\ln|x| + \frac{1}{2}\ln(x^2+1) - 3\ln|3x+2|\,.$$

Now differentiating both sides with respect to x, we get

$$\frac{f'(x)}{f(x)} = \frac{1}{3}\left(\frac{1}{x}\right) + \frac{1}{2}\left(\frac{2x}{x^2+1}\right) - 3\left(\frac{3}{3x+2}\right) = \frac{1}{3x} + \frac{x}{x^2+1} - \frac{9}{3x+2}.$$

Therefore

$$f'(x) = f(x)\left(\frac{1}{3x} + \frac{x}{x^2+1} - \frac{9}{3x+2}\right)$$

$$= \frac{x^{\frac{1}{3}}\sqrt{x^2+1}}{(3x+2)^3}\left(\frac{1}{3x} + \frac{x}{x^2+1} - \frac{9}{3x+2}\right).$$

Example 3.56 Find the derivative of the function $f\colon (0, +\infty) \to \mathbb{R}$ defined by

$$f(x) = x^x.$$

Solution:

For each $x > 0$, taking natural logarithm on both sides, we have

$$\ln f(x) = x\ln x\,.$$

Now differentiating both sides with respect to x, we get

$$\frac{f'(x)}{f(x)} = (1)(\ln x) + (x)\left(\frac{1}{x}\right) = \ln x + 1.$$

Therefore

$$f'(x) = f(x)(\ln x + 1) = x^x(\ln x + 1).$$

Remark 3.57 The following is a summary on the **rules of differentiation** and the **derivatives of elementary functions**.

Rules of Differentiation

The following equalities hold under the assumption that all the relevant derivatives exist:

(**Linearity I**) $(cf)'(x) = cf'(x)$ *for every* $c \in \mathbb{R}$ $\qquad \frac{d}{dx}(cu) = c\frac{du}{dx}$ *for every* $c \in \mathbb{R}$

(**Linearity II**) $(f \pm g)'(x) = f'(x) \pm g'(x)$ $\qquad \frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx}$

(**Product rule**) $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$ $\qquad \frac{d}{dx}(uv) = \frac{du}{dx}v + u\frac{dv}{dx}$

(**Quotient rule**) $\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$ $\qquad \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{\frac{du}{dx}v - u\frac{dv}{dx}}{v^2}$

(**Chain rule**) $(f \circ g)'(x) = f'\big(g(x)\big)g'(x)$ $\qquad \frac{dy}{dx} = \frac{dy}{dv}\frac{dv}{dx}$

(**Inverse**) $(f^{-1})'(x) = \frac{1}{f'\left(f^{-1}(x)\right)}$ $\qquad \frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$

Derivatives of Elementary Functions

The following differentiation formulas hold ***on open intervals****:*

(i) ***Polynomials and power functions***

$\frac{d}{dx}c = 0$ *for every* $c \in \mathbb{R}$ $\qquad \frac{d}{dx}x^r = rx^{r-1}$ *for every* $r \in \mathbb{R}$

(ii) ***Trigonometric functions***

$\frac{d}{dx}\sin x = \cos x$ $\qquad \frac{d}{dx}\tan x = \sec^2 x$ $\qquad \frac{d}{dx}\sec x = \sec x \tan x$

$\frac{d}{dx}\cos x = -\sin x$ $\qquad \frac{d}{dx}\cot x = -\csc^2 x$ $\qquad \frac{d}{dx}\csc x = -\csc x \cot x$

(iii) ***Exponential functions and logarithmic functions***

$\frac{d}{dx}e^x = e^x$ $\qquad \frac{d}{dx}a^x = a^x \ln a$ *for every* $a > 0$

$\frac{d}{dx}\ln x = \frac{1}{x}$ $\qquad \frac{d}{dx}\log_a x = \frac{1}{x \ln a}$ *for every* $a \in (0, +\infty) \setminus \{1\}$

(iv) ***Inverse trigonometric functions***

$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}$ $\qquad \frac{d}{dx}\arctan x = \frac{1}{1+x^2}$ $\qquad \frac{d}{dx}\operatorname{arcsec} x = \frac{1}{|x|\sqrt{x^2-1}}$

$\frac{d}{dx}\arccos x = \frac{-1}{\sqrt{1-x^2}}$ $\qquad \frac{d}{dx}\operatorname{arccot} x = \frac{-1}{1+x^2}$ $\qquad \frac{d}{dx}\operatorname{arccsc} x = \frac{-1}{|x|\sqrt{x^2-1}}$

Remark 3.58 It is very important to note that the **differentiation formulas** in (the bottom half of) Remark 3.57 are **valid on <u>open intervals</u> only**.

- For example, if $f:[0,1] \to \mathbb{R}$ is defined by $f(x) = x^2$ for $x \in [0,1]$, then the "power rule" only says that $f'(x) = 2x$ for $x \in (0,1)$. f is not differentiable at the end-points 0 and 1, because $f'_-(0)$ and $f'_+(1)$ do not exist as f is not defined for $x < 0$ or for $x > 1$.

As a result, when f is a **piecewise defined** function, we need to be careful when computing f'.

- On each **open interval** where f is elementary, we can still evaluate f' using **differentiation formulas** in Remark 3.57.
- However, at those points where the definitions of f on the two sides are different, we must go back to the basics and apply the **definition of derivative**.

Example 3.59 Find the derivative of the function $f:\mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 - \cos x & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases}.$$

Solution:

To find $f'(x)$, we need to consider the following **three** cases:

(i) For $x < 0$, (by the differentiation formulas) we have $f'(x) = \frac{d}{dx}(1 - \cos x) = \sin x$.

(ii) For $x > 0$, (by "power rule") we have $f'(x) = \frac{d}{dx}x = 1$.

We cannot deduce $f'(x) = 1$ **for** $x \geq 0$ directly from $f(x) = x$ for $x \geq 0$. Differentiation formulas are valid on **open intervals** only.

(iii) For $x = 0$, we apply the **definition of derivative**. We have

$$\lim_{x\to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0^+} \frac{x - 0}{x - 0} = \lim_{x\to 0^+} 1 = 1,$$

$$\lim_{x\to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0^-} \frac{(1 - \cos x) - 0}{x - 0} = \lim_{x\to 0^-} \frac{2\sin^2(x/2)}{x} = \lim_{x\to 0^-} \frac{\sin(x/2)}{x/2} \lim_{x\to 0^-} \sin\frac{x}{2}$$
$$= 1 \cdot 0 = 0.$$

Therefore $f'(0)$ does not exist.

In summary, the derivative $f':\mathbb{R}\setminus\{0\} \to \mathbb{R}$ is given by

$$f'(x) = \begin{cases} \sin x & \text{if } x < 0 \\ 1 & \text{if } x > 0 \end{cases}.$$

Remark 3.60 In Example 3.59, it may be tempting for us to just say that "for $x = 0$, we have

$$\lim_{x\to 0^-} f'(x) = \lim_{x\to 0^-} \sin x = 0 \qquad \text{and} \qquad \lim_{x\to 0^+} f'(x) = \lim_{x\to 0^+} 1 = 1$$

which are different, so $f'(0)$ does not exist". Although this argument leads to the correct conclusion, its reasoning is <u>**incorrect**</u>; what this argument really implies is that "$\lim_{x\to 0} f'(x)$ does not exist", instead of that "$f'(0)$ does not exist". $\lim_{x\to 0} f'(x)$ and $f'(0)$ are different in general.

Summary of Chapter 3

The following are what you need to know in this chapter in order to get a pass (a distinction) in this course:

- ✓ Definition of the **derivative** of a function
- ✓ Definition of **differentiability** at a point and on an interval
- ✓ **Geometric interpretation** of the derivative: Slopes of tangent lines
  - ⦿ **Equations of tangent lines and normal lines** to the graph of a function
- ✓ **Physical interpretation** of the derivative: Instantaneous rates of change
- ✓ Various **notations** of the derivative
- ✓ One-sided derivatives
- ✓ Differentiability implies continuity
- ✓ Non-differentiability: Discontinuity, corner point, vertical tangent line, no tangent line

- ✓ Rules of differentiation
  - ⦿ Sum and difference
  - ⦿ **Product rule and quotient rule**
  - ⦿ **Chain rule**
  - ⦿ Derivative of the **inverse**
- ✓ Derivatives of elementary functions
  - ⦿ **Polynomials** and **power** functions
  - ⦿ **Trigonometric** functions
  - ⦿ **Exponential** functions
  - ⦿ **Logarithmic** functions
  - ⦿ **Inverse trigonometric** functions
- ✓ Techniques of differentiation
  - ⦿ **Implicit differentiation**
  - ⦿ **Logarithmic differentiation**

- ✓ Rules and formulas of differentiation are **valid on open intervals only**
- ✓ Differentiability of **piecewise defined functions**